Asymptotic Expansion for Local Volatility Models

Dan Pirjol\textsuperscript{1}

\textsuperscript{1}J. P. Morgan, New York

20 Sept. 2011
Outline

- Local volatility in terms of implied volatility
- Implied volatility in terms of local volatility
  - The BBF expansion of the Dupire equation
  - Properties of the implied volatility in local vol models
- Normal implied volatility, and its small time expansion
- Illustrating the expansion on simple examples
- Summary and conclusions
[1] Jim Gatheral,
The Volatility Surface: A Practitioner's Guide

Analysis, Geometry and Modeling in Finance: Advanced Methods in Option Pricing

Local Volatility: Statics, Dynamics, and Probabilistic Interpretation

[4] H. Berestycki, J. Busca and I. Florent,
Asymptotics and Calibration of Local Volatility Models,

[5] V. Costeanu and D. Pirjol,
Asymptotic Expansion for the Normal Implied Volatility
Options

Options - financial contracts (derivatives) on financial assets.

Example

stock of publicly traded company $S(t)$
Call/Put option: the right to buy/sell the stock at predetermined price $K$

Payoff

\[
\text{Call} : \max (S(T) - K, 0) \\
\text{Put} : \max (K - S(T), 0)
\]
Some financial notation

- **Spot price of a stock**: the price of the stock today $S(t)$
- **Forward price of a stock today at some time $T$ in the future**: $F(t, T)$ is the price one would have to pay today to own one share of the stock at time $t$

Generally the forward price is larger than the spot because of interest rates. Explicitly

$$F(t, T) = S(t)e^{r(T-t)(T-t)}$$

where $r(t, T)$ is the interest rate for borrowing over time $T-t$ at time $t$
Risk-neutral pricing

The traditional approach in pricing theory is to say (much simplified)

- Assume that the spot price follows the real life process
  \[
  \frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t)dW_t
  \]

- If we can find a probability measure such that the spot price is a martingale, the process in this measure is driftless
  \[
  \frac{dS_t}{S_t} = \sigma(S_t, t)dW_t, \quad \mathbb{E}[S(T)|S(t)] = F(t, T)
  \]

- Then any payoff can be priced as expectation under this measure - risk-neutral measure

For simplicity, assume that the risk-free interest rate is zero.
Any payoff can be priced as an expectation in the risk-neutral measure.
Simple expression for option prices in the risk-neutral measure

$$\text{Call}(K, t) = \int_0^\infty dS \varphi(S, t)(S(t) - K)_+$$

$$\text{Put}(K, t) = \int_0^\infty dS \varphi(S, t)(K - S(t))_+$$

where $\varphi(x, t)$ is the spot price density at time $t$.
Also, the martingale condition expresses the forward price as an average of the spot price in the risk-neutral measure

$$F(0, T) = \int_0^\infty dS S(T) \varphi(S, T)$$
Distributional properties

The stock terminal distribution over long times (days, weeks) is approximatively log-normal

\[ S(T) = S(0) \exp(\sigma \sqrt{T} X - \frac{1}{2} \sigma^2 T), \quad X \sim N(0, 1) \]

**Advantage:** \( S(T) \) is strictly positive

Over short time intervals (\(< 1\) day, hours) a better approximation is a normal distribution

\[ S(T) = S(0) + \sigma \sqrt{T} X \]

Actually first considered in option theory by Bachelier (1912)
Pricing formulas

- Assuming log-normal distribution for $S(T)$, the option price is the Black-Scholes formula

$$C_{BS}(K, S_0, T, \sigma) = S_0 N(d_1) - K N(d_2)$$

$$d_{1,2} = \frac{1}{\sigma \sqrt{T}} \left( \log \frac{S_0}{K} \pm \frac{1}{2} \sigma^2 T \right)$$

- Assuming a normal distribution for $S(T)$, the option price is the Bachelier formula

$$C_N(K, T, \sigma) = (S_0 - K) N \left( \frac{S_0 - K}{\sigma \sqrt{T}} \right) + \frac{1}{\sqrt{2\pi}} \sigma \sqrt{T} e^{-\frac{(K - S_0)^2}{2\sigma^2 T}}$$
Probabilities are prices

We can recover risk-neutral probabilistic information from derivative prices. From basic probability we can determine the probability for event $A$ as

$$Pr(A) = \mathbb{E}[1_A]$$

But expectations correspond to prices of instruments in the risk-neutral measure, so we get for example

$$Pr(S(T) > K) = \mathbb{E}[1_{S(T) > K}]$$
Implied distributions

Knowing the vanilla option prices, we can recover considerable information about the risk neutral distribution

Consider the price of a call option

\[ C(K, t) = \int_{-\infty}^{\infty} dx (x - K) \varphi(x, t) \]

Taking one derivative we find

\[ \partial_K C(K, t) = -Pr(S(T) > K) = \int_{-\infty}^{\infty} dx \varphi(x, t) \]

The second derivative of the call price is precisely the risk neutral density

\[ \partial_K^2 C(K, t) = \varphi(K, t) \]
More complicated payoffs: European payoffs

The market prices of vanilla options give access to the risk neutral density of the asset $\varphi(S, t)$. In fact both are equivalent information.

Knowledge of $\varphi(S, t)$ allows us to price any ”European” payoff

**Definition**
European payoff $=$ payoff which depends only on the realization of $S(t)$ at one given time, but not e.g. on the joint distribution of $S(t_1), S(t_2), \cdots$.

The price of an instrument paying $f(S(t))$ at time $t$

$$\mathbb{E}[f(S(t))] = \int_0^{\infty} dx f(x) \varphi(x, t)$$
More complicated payoffs: path-dependent payoffs

**Definition**: *Path-dependent instrument* = payoff depends on the entire path of $S(t)$ with $t_1 \leq t \leq t_2$

Examples:

1. Barrier option (knock-out)
   
   $$\text{Payoff} = \max (S(T) - K, 0) 1_{S(t) < B, t < T}$$

2. Asian option (average option)
   
   $$\text{Payoff} = \max (I(T) - K, 0), \quad I(T) = \frac{1}{T} \int_0^T dsS(s)$$

Knowledge of the density $\varphi(S, t)$ at one time $t$ alone is not sufficient to price such instruments. The entire joint distribution of $S(t)$ is needed - the process for $S(t)$
Implied Volatility: Log-normal and Normal

Market option prices can be expressed in terms of the (log-normal) implied volatility $\Sigma(K, t) =$ the volatility at which the Black-Scholes price reproduces the option price

$$C(K, t) = C_{BS}(K, t, \Sigma(K, t))$$

The map $\sigma \rightarrow C_{BS}(K, \sigma, t)$ is invertible $\rightarrow$ can always find an implied volatility provided that

$$C(K, S_0, t) \geq S_0 - Ke^{-rt}$$

Using the Bachelier pricing formula one can define also a normal implied volatility $\Sigma_N(K, t)$ - more on this later

$$C(K, t) = C_N(K, t, \Sigma_N(K, t))$$
Implied Volatility

The map $\Sigma \rightarrow C_{BS}(K, \sigma)$ at given log-strike $x = \log \frac{S_0}{K}$ is invertible.

Figure 1: Normalised call ($\theta = 1$) option prices given by (2.2).

Jaeckel, By Implication
The volatility surface

The implied volatility $\Sigma(K, t)$ depends on the strike $K$ of the option - the smile.

The volatility surface $\Sigma(K, t)$ of the S&P500 index as of 6-Sep-2011.
What is the reason for the smile?

- **Behavioral causes**
  - Crash protection, fear of crashes. Anticipation of higher/lower prices in the future leads to higher vols for options with larger/smaller strikes
  - Expectation of changes in volatility over time
  - Support/resistance levels at various strikes

- **Structural causes: violations of the Black-Scholes assumptions**
  - Inability of hedging continuously
  - Transaction costs
  - Price-dependent volatility $\sigma(S, t)$
  - Stochastic volatility

Dan Pirjol

Asymptotic Expansion for Local Volatility Models
Local volatility model

Is it possible to find a one-dimensional stock process such that it reproduces a given implied volatility surface $\Sigma(K,t)$?

$$dS(t) = \sigma_D(S,t)S(t)dW(t) + \mu(t)S(t)dt$$

We can use the resulting process to price any European payoffs $f(S(t))$.

The process $S(t)$ will give also a possible joint distribution of $S(t_i)$ at different times $t_i$. We can also use it to price path-dependent products, but it is not unique $\rightarrow$ possibly wrong dynamics!

Problem: find $\sigma_D(S,t)$ (the local volatility) and $\mu(t)$ such that the process $S(t)$ reproduces the density $\varphi(S,t)$ implied from vanilla option prices.

The solution to this problem was found by Dupire and Derman and Kani, and is known as the local volatility model.
Finding the parameters of the local volatility process

Use the Fokker-Planck equation

\[ \partial_t \varphi(x, t) = -\partial_x [\mu(t)x \varphi(x, t)] + \frac{1}{2} \partial_x^2 [x^2 \sigma^2_D(x, t) \varphi(x, t)] \]

The drift can be simply expressed in terms of the forward price \( F(t, T) \). Recall that in the risk-neutral measure

\[ F(t, T) = \int_0^\infty dx x \varphi(x, T) \]

Taking a derivative wrt time

\[ \partial_T F(t, T) = \int_0^\infty dx x \left( -\partial_x [\mu(t)x \varphi(x, t)] + \frac{1}{2} \partial_x^2 [x^2 \sigma^2_D(x, t) \varphi(x, t)] \right) = -\mu(T) F(t, T) \]

The drift \( \mu(T) \) is the time derivative of \( \log F(t, T) \).
Finding the local volatility $\sigma_D(S, t)$

Take a derivative wrt time of the call price

$$\partial_T C(K, T) = \int_K^\infty dx (x - K) \partial_T \varphi(x, T)$$

$$\int_K^\infty dx (x - K) \left( - \partial_x [\mu(t)x \varphi(x, t)] + \frac{1}{2} \partial^2_x [x^2 \sigma^2_D(x, t) \varphi(x, t)] \right)$$

After integrating by parts and dropping boundary terms, the two integrals give

$$\#1 = \int_K^\infty dx (x - K) \partial_x [\mu(t)x \varphi(x, t)] = -\mu(T) \int_K^\infty dxx \varphi(x, T)$$

$$= -\mu(T)(1 - K \partial_K) C(K, T)$$
Finding the local volatility $\sigma_D(S, t)$

Take a derivative wrt time of the call price

$$\partial_T C(K, T) = \int_{\infty}^{\infty} dx(x - K)\partial_T \varphi(x, T)$$

$$\int_K^{\infty} dx(x - K) \left( -\partial_x [\mu(t)x\varphi(x, t)] + \frac{1}{2}\partial_x^2 [x^2\sigma_D^2(x, t)\varphi(x, t)] \right)$$

After integrating by parts and dropping boundary terms, the two integrals give

$$#2 = \int_K^{\infty} dx(x - K)\partial_x^2 [x^2\sigma_D^2(x, t)\varphi(x, t)]$$

$$= -K^2\sigma_D^2(K, T)\varphi(K, T)$$
Combining everything together one finds the equation satisfied by the call option prices

**Dupire equation**

\[ \partial_T C(K, T) = \frac{1}{2} K^2 \sigma_D^2(K, T) \partial_K^2 C(K, T) + \mu(T)(1 - K \partial_K)C(K, T) \]

From this we can determine the local volatility \( \sigma_D(S, T) \) provided that we know the call option prices \( C(K, T) \) with arbitrary strike \( K \) and maturity \( T \)
In the local volatility model the stock price $S(t)$ follows the process

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma_D(S, t)dW(t)$$

with the drift $\mu(t)$ and local volatility $\sigma_D(S, t)$ given by

$$\mu(T) = -\partial_T \log F(0, T)$$

$$\sigma_D^2(K, t) = 2 \frac{\partial_T C(K, T) - \mu(T)(1 - K\partial_K)C(K, T)}{K^2\partial^2_K C(K, t)}$$

The process for $S(t)$ can be plugged into a Monte Carlo simulation and used for pricing non-vanilla products.
The BBF formulation of the Dupire equation

Alternative formulation: replace the call prices $C(K, T)$ with the implied volatility $\Sigma(K, T)$

$$
\sigma_D^2(x, t) = \frac{\Sigma^2(x, T) + T \partial_T \Sigma^2(K, T) + \mu(T) T \partial_x \Sigma^2(x, T)}{(1 - \frac{x}{\Sigma} \partial_x \Sigma)^2 + \Sigma T \partial_x^2 \Sigma - \frac{1}{4} \sigma^2 T^2 (\partial_x \Sigma)^2}
$$

Change of variable: $K \rightarrow x = \log \frac{K}{S(0)}$ (log-strike)

Practical considerations

In practice we don’t have a smooth and continuous call option price surface $C(K, T)$, but just a few quotations with strikes $K$ around the at-the-money point $K = F(t, T)$. Interpolation of the implied volatility surface $\Sigma(K, t)$ is needed away from the ATM point.

Issues:

- The interpolation may produce a negative $\sigma_D^2(K, t)$. This signals arbitrage, and must be avoided by choosing a different functional dependence for the implied volatility far away from the ATM point.

- The implied volatility can grow at most like $\Sigma^2(x, t) \to c_{L,R} |x|$, where the constants $c_{L,R}$ depend on the highest positive and negative moment of the distribution $\varphi(x, t)$ which is still finite.

Implied volatility from local volatility

Consider the inverse problem: given a local volatility function $\sigma_D(S, t)$, find the corresponding implied volatility $\Sigma(K, t)$

This could be motivated by the use of a certain parametric representation for the local volatility. Easier to fit the observed implied volatility to a simple functional form for $\sigma_D(S, t)$

Several possible approaches:

1. Solve the Dupire equation for $C(K, t)$ with the initial condition $C(K, 0) = \max(S_0 - K, 0)$, and then solve for the implied volatility $\Sigma(K, t)$

2. Solve the BBF equation for $\Sigma(K, t)$ in a small-time expansion

3. Most likely path method (Gatheral)
Simple limiting case: local volatility depends only on time

Assume that the local volatility function $\sigma_D(t)$ depends only on time. The equation for the implied volatility reads

$$\sigma_D^2(t) = \partial_t (t\Sigma^2(t))$$

Can be easily solved

$$\Sigma_D^2(t) = \frac{1}{t} \int_0^t ds \sigma_D^2(s)$$

**Corollary**

The implied volatility $\Sigma^2(t)$ is the time average of the local volatility $\sigma_D^2(t)$
Finding the implied volatility

Approach 1. Two-step process:

- Solve the Dupire equation for the call option price $C(K, T)$

$$
\partial_T C(K, T) = \frac{1}{2} K^2 \sigma^2_D(K) \partial_K^2 C(K, T) + \mu(T)(1 - K \partial_K)C(K, T)
$$

with the initial condition $C(K, 0) = (S_0 - K)_+$

Heat equation with space-dependent conductivity $\sigma_D^2(x)$, can be solved in a small-time expansion using e.g. heat kernel methods

- Solve for the implied volatility $\Sigma(K, T)$

Complicated by the presence of an essential singularity in the small-time expansion of the Black-Scholes price (unless $K = S_0$)

$$
C_{BS}(K, S_0, t) - (S_0 - K)_+ = \frac{\sqrt{S_0 K}}{\sqrt{2\pi x^2}} (\sigma^2 t)^{3/2} \exp\left(-\frac{x^2}{2\sigma^2 t}\right) + O(t^{5/2})
$$
Approach 1: solving the Dupire equation

Numerical approach. Easy to solve e.g. in Mathematica

```mathematica
solver[S0_, SpotVol_, VolVol_, rho_, T_, K_, npts_] :=
Module[{grid, soln, ans, a, Smin, Smax},
  Smin = S0 - 4 SpotVol Sqrt[T];
  Smax = S0 + 4 SpotVol Sqrt[T];
  a[S_] := LocalVol[S, SpotVol, VolVol, S0];
  grid = Smin + Table[(Smax - Smin)i/npts, {i, 0, npts}];
  soln = V /. NSolve[
    {D[V[S, t], t] == 1/2 a[S]^2 D[V[S, t], {S, 2}], V[S, 0] ==
      Max[S - K, 0], V[Smin, t] == 0, V[Smax, t] == Max[Smax - K, 0],
      V, {S, Smin, Smax}, {t, 0, T},
  ans = soln[S0, T];
  Return[ans];
]
```
Approach 2: small-time expansion for the implied volatility

Solve the Dupire equation for the implied volatility $\Sigma(K, t)$ in a small-time expansion

Expand the local volatility in time

$$\sigma_D(S, t) = \sigma_{D,0}(S) + \sigma_{D,1}(S)t + \sigma_{D,2}(S)t^2 + \cdots$$

Assume that the solution can be also expanded in powers of time

$$\Sigma(K, t) = \Sigma_0(K) + \Sigma_1(K)t + \Sigma_2(K)t^2 + \cdots$$

At zeroth order in $t$ we get the equation

$$\sigma_{D,0}(x) = \frac{\Sigma_0(x)}{1 - \frac{x}{\Sigma_0(x)} \partial_x \Sigma_0(x)}$$
Small-time asymptotics of the implied volatility

The equation for the implied volatility can be solved in closed form at zeroth order in the time expansion

*The BBF formula*

\[ \Sigma_0(x) = \frac{x}{\int_0^x \frac{dy}{\sigma_{D,0}(y)}} \text{,} \quad x = \log \frac{K}{S_0} \]

**Theorem**

*The implied volatility is the harmonic average of the local volatility.*
Small-time asymptotics of the implied volatility

A more intuitive representation of the implied volatility can be obtained by approximating the integral in the denominator of the BBF formula with the rectangular rule

\[
\int_0^x \frac{dy}{\sigma_{D,0}(y)} \simeq x \frac{1}{\sigma_D(x/2)} = \frac{\log \frac{K}{S_0}}{\sigma_D(\sqrt{KS_0})}
\]

This gives

\[
\Sigma_0(K) \simeq \sigma_D(\sqrt{KS_0})
\]

**Corollary**

*The implied volatility at strike K is approximately given by the local volatility at the geometric average of the strike and spot*
The skew of the implied volatility

Taking derivatives with respect to the log-strike \( x = \log \frac{K}{S_0} \) at the at-the-money (ATM) point \( x = 0 \) we find

\[
\partial_x \Sigma(0) = \frac{1}{2} \partial_x \sigma_D(0)
\]

**Corollary**

*The ATM slope of the implied volatility is approximately one half of that of the local volatility at the same point*

The implied volatility surface is less skewed than the local volatility surface
Limitations of the BBF formula

The leading asymptotic implied volatility $\Sigma_0(K)$ for $K > S_0$ is insensitive to the shape of the local volatility $\sigma_D(K)$ in the region $K < S_0$.

The continuous and dashed local volatility functions $\sigma_D(K)$ produce the same asymptotic implied volatility $\Sigma_0(K)$ for $K > S_0$.

The reason for this behaviour is that although the Dupire equation for $\Sigma(K, t)$ is second order in strike, it becomes first order after expanding in time.
Example: the CEV model

The Constant Elasticity of Variance (CEV) model is a popular smile model. It is defined by the diffusion

$$\frac{dS(t)}{S(t)} = a S^{\beta-1}(t) dW(t)$$

Limiting cases:
- $\beta = 1$ log-normal diffusion, no smile
- $\beta = 0$ normal diffusion

The BBF formula gives the leading order asymptotics for the implied volatility

$$\Sigma_0(K) = a (1 - \beta) \frac{\log \frac{K}{S_0}}{K^{1-\beta} - S_0^{1-\beta}}$$

up to terms of order $O(t)$.
Consider the CEV model with $\beta = \frac{1}{2}$ for which we have closed form results for the terminal stock distribution.

$$a = 0.2, \ T = 1 \text{ (red), } T = 10 \text{ (blue)}$$
For reasonably regular local volatility \( \sigma_D(S, t) \), the implied volatility can be found from a small-time expansion of the Dupire equation

\[
\Sigma(K, t) = \Sigma_0(K) + \Sigma_1(K)t + \Sigma_2(K)t^2 + \cdots
\]

Simple closed form formula (BBF) for the zero-th order in the small time expansion \( \Sigma_0(K) \)

Relatively simple formula also for the linear term \( \Sigma_1(K) \), higher order terms \( \Sigma_{k>2}(K) \) more complicated but straightforward to obtain - can be expressed by quadratures, which can be solved numerically
**Normal implied volatility**

**Definition**
Normal implied volatility $\Sigma_N(K, t) =$ the value of the volatility which, when inserted into the Bachelier formula, reproduces a given option price

$$C(K, t) = C_N(K, t, \Sigma_N(K, t))$$

where the Bachelier pricing formula is given by

$$C_N(K, t, \sigma) = (S_0 - K)N\left(\frac{S_0 - K}{\sigma \sqrt{t}}\right) + \frac{1}{\sqrt{2\pi}} \sigma \sqrt{t} e^{-\frac{(K-S_0)^2}{2\sigma^2 t}}$$

Superficially very similar to the log-normal implied volatility.
Why normal implied volatility?

Some assets are not necessarily positive definite, so a log-normal distribution is not appropriate (too restrictive).

*Examples:*

- Interest rates for small rates. In the regime of small interest rates (post 2008), forward rates can become negative.
- The spread between two assets can take either sign.

The normal volatility is commonly used for quoting swaption volatilities (yield volatility, bp volatility). Historically, it is more stable against changes in the level of the interest rates.
Can one reformulate the Dupire equation in terms of normal implied volatilities?

This is indeed possible, with several modifications.

The process for $S(t)$ takes the form

$$dS(t) = \sigma_D(S, t)dW(t) + \mu(t)dt$$

where the parameters of the diffusion are

$$\mu(T) = \partial_T F(0, T)$$
$$\sigma_D^2(K, t) = 2\frac{\partial_T C(K, T) + \mu(T)\partial_K C(K, T)}{\partial_K^2 C(K, t)}$$
Normal implied volatility from the local volatility

Replace the call prices $C(K, T)$ with the implied normal volatility $\Sigma_N(K, T)$

The Dupire equation in BBF formulation

$$
\sigma_D^2(y, t) = \frac{\Sigma_N^2(y, T) + T \partial_T \Sigma_N^2(y, T) + \mu(T) T \partial_y \Sigma_N^2(y, T) \left(1 - \frac{y}{\Sigma_N} \partial_y \Sigma_N\right)^2 + \Sigma_N T \partial_y^2 \Sigma_N}{(1 - \frac{y}{\Sigma_N} \partial_y \Sigma_N)^2 + \Sigma_N T \partial_y^2 \Sigma_N}
$$

Change of variable: $K \rightarrow y = K - S(0)$

Note the similarity of the equation with that for the log-normal case - although in a different independent variable. The $O(T^2)$ term in the denominator is absent here.

The solutions are identical at $O(T^0, T^1)$, up to different independent variables.
Assume for simplicity a time-homogeneous local volatility $\sigma_D(S)$

The Dupire equation is the same as in the log-normal case up to corrections of $O(t^2)$, although in a different variable $y = K - S_0$ vs $x = \log(K/S_0)$ so we have that

$$
\Sigma_{N,0}(y) = \frac{K - S_0}{\int_{S_0}^{K} \frac{dy}{\sigma_D(y)}}
$$

The short-time asymptotics of the normal implied volatility is also given by the harmonic average of the local volatility

$$
\Sigma_N(K, t) = \Sigma_{N,0}(K) + \Sigma_{N,1}(K)t + \Sigma_{N,2}(K)t^2 + \cdots
$$
Asymptotics of the normal implied volatility

The zeroth order result for the normal implied volatility is simply related to the log-normal volatility

\[ \Sigma_{N,0}(K) = \left( K - S_0 \right) \log \frac{K}{S_0} \Sigma_0(K) \]
Higher order corrections

Higher order terms in the time expansion can be also found straightforwardly. The correction $\Sigma_{N,j}(K)$ of $O(t^j)$ to the normal implied volatility satisfies the first order ODE

$$\frac{2y}{1 - \frac{y}{\Sigma_{N,0}} \frac{\partial_y \Sigma_{N,0}}{\Sigma_{N,0}} (\frac{\Sigma_{N,j}(y)}{\Sigma_{N,0}(y)})'} + (2j + 1) (\frac{\Sigma_{N,j}(y)}{\Sigma_{N,0}(y)}) + H_j(\Sigma_{N,k<j}(y), y) = 0$$

where $H_j$ depends only on lower order terms in the small-time expansion $\Sigma_{N,k<j}(y)$. The sequence of equations can be solved by a recursion, starting with the $O(t)$ term.

This ODE can be integrated; there is a unique solution which is finite at $y = 0$ given by

$$\Sigma_{N,j}(y) = -\Sigma_{N,0}(y) \left( \frac{\Sigma_{N,0}(y)}{y} \right)^{j+1} \int_0^y \frac{z^j}{2\sigma_D(z)\Sigma_{N,0}^j(z)} H_j(z)$$
First order correction $O(t)$

The first order correction to the normal implied volatility can be found in closed form

$$
\Sigma_{N,1}(y) = \frac{\Sigma_{N,0}^2(y)}{y^2} \left( - \frac{1}{2} \log \frac{\Sigma_{N,0}^2(y)}{\sigma_D(y)\Sigma_{N,0}(y)} + \mu_0 \int_{S_0}^{K} dz \left( \frac{1}{\Sigma_{N,0}(z)} - \frac{1}{\sigma_D(z)} \right)^2 \right)
$$

Simple expression for $K = S_0$ (at-the-money)

$$
\Sigma_{N,1}(0) = \frac{1}{24} \sigma_D(0) [2\sigma_D(0)\sigma_D''(0) - (\sigma_D'(0))^2].
$$
How well does it work?

Example: shifted log-normal model

\[ dS(t) = (\sigma_0 + 2bS(t))dW(t) \]

\[ S_0 = 3\%, \sigma_0 = 0.8\%, b = 0.1 \text{ and } T = 10 \text{ (left), } T = 30 \text{ (right)} \]

Black: exact, dashed red: \( \Sigma_{N,0}(K) \), blue: \( \Sigma_{N,0}(K) + \Sigma_{N,1}(K)T \).
Local volatility with discontinuous derivatives

Example: linear local volatility with discontinuous derivative at \( S = S_0 \)

\[
\sigma_D(y) = \begin{cases} 
\sigma_0 + 2b_L y, & y < 0 \\
\sigma_0 + 2b_R y, & y > 0 
\end{cases}
\]

\[
b_L = \begin{cases} 
0.1 \text{ (black)} \\
0 \text{ (blue)} \\
-0.1 \text{ (red)} 
\end{cases}
\]

\[
b_R = 0.1
\]

\[
S_0 = 3\%, \sigma_0 = 0.8\%, T = 10
\]

Solid curves: numerical solution of the Dupire equation
Dashed curves: leading asymptotic result \( \Sigma_{N,0}(K) \)
$b_R = -b_L$: exact solution

For $b_R = -b_L \equiv b > 0$ the exact solution of the model is available (Karatzsas, Shreve)
The ATM ($K = S_0$) normal implied volatility can be written as

\[
\Sigma_N(0) = \Sigma_{N1}(0, t) + \Sigma_{N2}(0, t)
\]
\[
\Sigma_{N1}(0, t) = \sigma_0 \sqrt{\frac{\pi}{2}} \frac{1}{b \sqrt{t}} \text{Erf}(\frac{b \sqrt{t}}{\sqrt{2}})
\]
\[
\Sigma_{N2}(0, t) = \frac{1}{2} \sigma_0 e^{-\frac{1}{2} b^2 t}
\]
\[
+ \frac{1}{2} \sigma_0 \sqrt{\frac{\pi}{2}} \left[ b \sqrt{t} + b \sqrt{t} \text{Erf}(\frac{b \sqrt{t}}{\sqrt{2}}) - \frac{1}{b \sqrt{t}} \text{Erf}(\frac{b \sqrt{t}}{\sqrt{2}}) \right]
\]

▶ $\Sigma_{N1}(K, t)$ is the normal implied volatility corresponding to $\sigma_D(y) = \sigma_0 + 2by$

▶ $\Sigma_{N2}(K, t)$ gives the effect of the “bend” at $y < 0$
\( b_R = -b_L : \) small time expansion

The small-time expansion of the exact solution contains a \( O(t^{1/2}) \) term

\[
\Sigma_{N2}(0, t) = \sigma_0 \left( \frac{1}{2} \sqrt{\pi} b \sqrt{t} + \frac{1}{3} b^2 t - \frac{1}{30} b^4 t^2 + \frac{1}{280} b^6 t^3 + \cdots \right)
\]

Such a term is not present in the usual time expansion of the Dupire equation!

- **Black curves:** numerical solution of the Dupire equation
- **Red dashed curve:** \( \Sigma_{N,0}(K) \)
- **Solid blue curve:** \( \Sigma_{N,0}(K) + \Sigma_{N,1}(K)t \)

\( S_0 = 3\%, \sigma_0 = 0.8\%, b = 0.1 \)
Observation

The leading asymptotic result $\Sigma_{N,0}(K)$ is only sensitive to the shape of the local volatility $\sigma_D(S)$ in the region $(S_0, K)$

$$
\begin{align*}
\Sigma_{N,0}(0) &= \begin{cases} 
2b_L \frac{K-S_0}{\log(1+\frac{2b_L}{\sigma_0} (K-S_0))} & K < S_0 \\
2b_R \frac{K-S_0}{\log(1+\frac{2b_R}{\sigma_0} (K-S_0))} & K > S_0
\end{cases}
\end{align*}
$$

The leading asymptotic normal implied volatility $\Sigma_{N,0}(K)$ for $K > S_0$ is insensitive to the shape of the local volatility $\sigma_D(K)$ with $K < S_0$

This persists also for the higher order terms $\Sigma_{N,i}(K)$!

The underlying reason for this seems to be that although the Dupire equation is second order in strike, it becomes first order after expanding in time.
Non-analytic local volatility

The presence of a $O(t^{1/2})$ term in the small-time expansion of the implied volatility is generic for non-analytic local volatility with a discontinuous derivative.

Using a perturbation expansion for the solution of the Dupire equation with a local volatility function $\sigma_D(y)$ which has a discontinuous derivative at the ATM point

$$\partial_y(\sigma_D^2(+\varepsilon)) - \partial_y(\sigma_D^2(-\varepsilon)) = (\Delta b)$$

one can show that the small-time expansion of the normal implied volatility contains a $O(\sqrt{t})$ term, which is proportional with $\Delta b$

$$\Sigma_N(0, t) = \bar{\Sigma}_N(0, t) + \frac{1}{16} \sqrt{\frac{\pi}{2}} \sigma_D(0, t) \sqrt{t} (\Delta b)$$

This term dominates numerically the normal implied volatility near the ATM point.
Summary

- The BBF expansion can be extended also to the problem of finding the normal implied volatility.
- Systematic solution to the problem of finding the normal implied volatility $\Sigma_N(K,t)$ for a given local volatility model $(\sigma_D(K,t), \mu(t))$ in a small-time expansion

$$\Sigma_N(K,t) = \Sigma_{N,0}(K) + \Sigma_{N,1}(K)t + \Sigma_{N,2}(K)t^2 + \cdots$$

- The leading term $\Sigma_{N,0}(K)$ is given by a result similar to the BBF formula for the log-normal implied vol, although in a different variable $y = K - S_0$ vs $x = \log(K/S_0)$.
- Relatively simple formula also for the linear term $\Sigma_1(K)$, higher order terms $\Sigma_{k>2}(K)$ more complicated but straightforward to obtain - are expressed by quadratures, can be solved numerically.
Summary

- The small-time expansion of the implied volatility requires special care if the local volatility has discontinuous derivatives. Additional terms may be required.
- A discontinuous derivative for $\sigma_D(K, t)$ introduces a $O(\sqrt{t})$ term in the implied volatility expansion, which is proportional with the jump of the derivative.