Regime Switching Models and Risk Management Tools

John Liechty
Penn State University
Broad Application for Regime Switching Tools
Sudden Change in Dynamics

- Change Point or Regime Shift
- Change is unobserved, dynamics are observed

\[ y_t = f(y|\theta, D_t, \mathcal{F}_t) \]

\[ \theta = \text{Parameters} \]
\[ D_t = \text{Hidden Markov Process} \]
\[ \mathcal{F}_t = \text{Information Filtration} \]
Markov Processes

- Marked Point Processes vs.
- Hidden Markov Chains
Markov Processes

• Marked Point Processes vs. $y_t = f(y|\theta, D_t, F_t)$
• Hidden Markov Chains

$\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$

$\theta_1, \theta_3, \theta_2, \theta_3, \theta_1, \theta_2, \theta_1, \theta_3$
Statistical Inference

- Bayesian Approach, Mixture of Distributions
- Break up the data set into ‘consistent subsets’ and fit different statistical models to each subset
Statistical Inference

- Bayesian Approach, Mixture of Distributions
- Time structure helps identify ‘consistent subsets’
Finance Example

• Market Stress: Volatility and Correlations Increase

• Regime Shifting Covariance Structures:
  – MV Normal Returns
  – Stochastic Volatility Models – Regime Shifting
  – Correlation Structure – Regime Shifting
Finance Example

Bank of America Historic Stock Price (log return = \log(Z_{t+1}/Z_t))
Finance Example

GS

Financial Crisis

GSK

BOA

GOOGLE
# Finance Example


<table>
<thead>
<tr>
<th></th>
<th>GS</th>
<th>GOOG</th>
<th>GSK</th>
<th>BOA</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>1.0000</td>
<td>0.0846</td>
<td>-0.0848</td>
<td>0.7355</td>
</tr>
<tr>
<td>GOOG</td>
<td>*</td>
<td>1.0000</td>
<td>0.2009</td>
<td>-0.2017</td>
</tr>
<tr>
<td>GSK</td>
<td>*</td>
<td>*</td>
<td>1.0000</td>
<td>-0.4799</td>
</tr>
<tr>
<td>BOA</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Correlation Increased

## Stocks Empirical Correlation Matrix After Oct, 2007

<table>
<thead>
<tr>
<th></th>
<th>GS</th>
<th>GOOG</th>
<th>GSK</th>
<th>BOA</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>1.0000</td>
<td>0.6364</td>
<td>0.6673</td>
<td>0.8423</td>
</tr>
<tr>
<td>GOOG</td>
<td>*</td>
<td>1.0000</td>
<td>0.3586</td>
<td>0.4586</td>
</tr>
<tr>
<td>GSK</td>
<td>*</td>
<td>*</td>
<td>1.0000</td>
<td>0.7403</td>
</tr>
<tr>
<td>BOA</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Model of Observed Returns

A portfolio of $p$ assets, $Y_t = H_t^{1/2}U_t$  

log return: $Y_t = \log(Z_t/Z_{t-1})$

where $U_t$ is i.i.d $(0, I_p)$ process.

$H_t = S_t \Gamma_t S_t$

$L = -\frac{1}{2} \sum_{t=1}^{T} (p \cdot \log(2\pi) + \log(|H_t|) + Y_t' H_t^{-1} Y_t),$

$= -\frac{1}{2} \sum_{t=1}^{T} (p \cdot \log(2\pi) + \log(|S_t \Gamma_t S_t|) + Y_t' S_t^{-1} \Gamma_t S_t^{-1} Y_t),$

$= -\frac{1}{2} \sum_{t=1}^{T} (p \cdot \log(2\pi) + 2 \log(|S_t|) + \log(|\Gamma_t|) + \tilde{U}_t' \Gamma_t \tilde{U}_t).$

$\tilde{U}_t = [\tilde{u}_{1,t}, \ldots, \tilde{u}_{K,t}]'$ is a zero-mean process with covariance matrix $\Gamma_t$
Stochastic Volatility

Stochastic Volatility Model (Kim 1994, Pereira 2007)

\[ y_t = \exp\left(\frac{h_t}{2}\right) \varepsilon_t \]

\[ h_{t+1} = \mu_{D_t} + \phi_{D_t} (h_t - \mu_{D_t}) + \tau_{D_t} \eta_t \]

\[ h_1 \sim N\left(\mu_{D_1}, \frac{\tau_{D_t}^2}{1 - \phi_{D_t}}\right) \]

Mean level of log volatility

Re-parameterization of $\phi'$, $-1 < \phi < 1$, persistency

$y_t$ is mean corrected log return, $y_t = \log(\frac{z_t}{z_{t-1}})$ – univariate

($\mu$, $\phi$, $\tau$) define the volatility structure
Multiple Hidden Markov Chains

One hidden regime process $D^c$ driving the correlation structure; $p$ hidden regime processes driving the individual variance of each asset ($p$ assets)

$$H_t = \begin{bmatrix}
    s_{1t,D_t} & 0 & 0 & 0 \\
    0 & s_{2t,D_{2t}} & 0 & 0 \\
    0 & 0 & \cdots & 0 \\
    0 & 0 & 0 & s_{pt,D_{pt}} \\
\end{bmatrix}
\begin{bmatrix}
    1 & \rho_{1,t,D_{Ct}} & \cdots & \rho_{1,pt,D_{Ct}} \\
    \rho_{2,t,D_{Ct}} & 1 & \cdots & \rho_{2,pt,D_{Ct}} \\
    \cdots & \cdots & \cdots & \cdots \\
    \rho_{p1,t,D_{Ct}} & \rho_{n2,t,D_{Ct}} & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
    s_{1t,D_t} & 0 & 0 & 0 \\
    0 & s_{2t,D_{2t}} & 0 & 0 \\
    0 & 0 & \cdots & 0 \\
    0 & 0 & 0 & s_{pt,D_{pt}} \\
\end{bmatrix}$$

$$\Gamma_t = \sum_{k=1}^{K^C} I(D^C_{tk} = n) \Gamma_n$$

$$y_{pt} = \exp(h_{pt} / 2) \xi_t$$

$$h_{p,t+1} = \mu_{D_{pt}} + \phi_{D_{pt}} (h_{pt} - \mu_{D_{pt}}) + \tau_{D_{pt}} \eta_t$$
Common Correlation Model

\[ \Gamma = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} \]

\[ \rho_{ij} \sim N(\delta_{ij}, \nu^2) \]
Regime Shifting Correlation

\[ \Gamma_t = \sum_{k=1}^{K_c} I(D_t^c = k) \Gamma_k \]

\[ \Gamma_1 = \begin{bmatrix} \rho_{11}^1 & \cdots & \rho_{1p}^1 \\ \vdots & \ddots & \vdots \\ \rho_{p1}^1 & \cdots & \rho_{pp}^1 \end{bmatrix} \]

\[ \Gamma_2 = \begin{bmatrix} \rho_{11}^2 & \cdots & \rho_{1p}^2 \\ \vdots & \ddots & \vdots \\ \rho_{p1}^2 & \cdots & \rho_{pp}^2 \end{bmatrix} \]

\[ D^c \]

\[ K_c = 2 \]
The 1987 Crash

US

UK

GERMANY

JAPAN
The 1987 Crash

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>GERMANY</th>
<th>JAPAN</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Empirical Covariance</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US</td>
<td>1.077</td>
<td>0.421</td>
<td>0.693</td>
<td>0.137</td>
</tr>
<tr>
<td>UK</td>
<td>*</td>
<td>0.943</td>
<td>1.006</td>
<td>0.351</td>
</tr>
<tr>
<td>GERMANY</td>
<td>*</td>
<td>*</td>
<td>2.028</td>
<td>0.447</td>
</tr>
<tr>
<td>JAPAN</td>
<td>*</td>
<td>*</td>
<td></td>
<td>1.901</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>GERMANY</th>
<th>JAPAN</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Empirical Correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US</td>
<td>1.0</td>
<td>0.418</td>
<td>0.469</td>
<td>0.096</td>
</tr>
<tr>
<td>UK</td>
<td>*</td>
<td>1.0</td>
<td>0.727</td>
<td>0.262</td>
</tr>
<tr>
<td>GERMANY</td>
<td>*</td>
<td>*</td>
<td>1.0</td>
<td>0.227</td>
</tr>
<tr>
<td>JAPAN</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Correlation

Highest Correlation Regime

US

UK

GERMANY

JAPAN

International Indices, Regime
Correlation

\[
\Gamma = \begin{bmatrix}
\rho_{11} & \rho_{12} & \rho_{13} \\
\rho_{21} & \rho_{22} & \rho_{23} \\
\rho_{31} & \rho_{32} & \rho_{33}
\end{bmatrix}
\]

\[
\rho_{ij} \sim N(\delta_{ij}, \nu^2)
\]
Stochastic Volatility

<table>
<thead>
<tr>
<th>Asset</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \tau_1^2 )</th>
<th>( \tau_2^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>-1.12(0.05)</td>
<td>0.32(0.04)</td>
<td>0.59(0.08)</td>
<td>0.81(0.07)</td>
<td>0.29 (0.08)</td>
<td>0.13(0.06)</td>
</tr>
<tr>
<td>UK</td>
<td>-1.22(0.04)</td>
<td>-0.068(0.06)</td>
<td>0.34(0.07)</td>
<td>0.74(0.09)</td>
<td>0.26(0.06)</td>
<td>0.20(0.07)</td>
</tr>
<tr>
<td>GERM</td>
<td>-0.61(0.05)</td>
<td>0.61(0.05)</td>
<td>0.44(0.04)</td>
<td>0.77(0.07)</td>
<td>0.26(0.06)</td>
<td>0.18(0.06)</td>
</tr>
<tr>
<td>JAPAN</td>
<td>-0.41(0.06)</td>
<td>0.69(0.04)</td>
<td>0.44(0.08)</td>
<td>0.83(0.09)</td>
<td>0.39(0.05)</td>
<td>0.11(0.06)</td>
</tr>
</tbody>
</table>

\[ y_t = \exp(h_t / 2) \varepsilon_t \]

\[ h_{t+1} = \mu_{D_t} + \phi_{D_t} (h_t - \mu_{D_t}) + \tau_{D_t} \eta_t \]

\[ h_1 \sim N(\mu_{D_1}, \frac{\tau_{D_t}^2}{1 - \phi_{D_t}^2}) \]

Mean level of log volatility

Re-parameterization of \( \phi' \), -1<\( \phi \)<1, persistency
Stochastic Volatility
Stochastic Volatility
Applications

• Value at Risk (VaR)
  – Measure of risk in firm’s portfolio, \( r_{p,t+1} = w^T r_{t+1} \)
  – Portfolio weights, \( w \), and asset returns, \( r_{t+1} \)

• Portfolio Allocation
  – How much should a firm hold of each asset; choosing, \( w \)
Value at Risk

- Find VaR, such that

\[ \alpha = Pr(r_{p,t+1} < \text{VaR}) \]

\[ r_{p,t+1} = w^T r_{t+1} \]

\[ r_{t+1} \sim N(\mu, H_t) \]
Value at Risk

• Find VaR, such that

\[ \alpha = Pr \left( r_{p,t+1} < VaR \right) \]

\[ r_{p,t+1} \sim N \left( \mu_p = w^T \mu, \sigma^2_{p,t} = w^T H_t w \right) \]

\[ \alpha = Pr \left( \frac{r_{p,t+1} - \mu_p}{\sigma_{p,t}} < Z_\alpha \right) \]

• E.g. \[ \alpha = 0.025 \Rightarrow Z_\alpha = -1.96 \]

\[ 0.025 = Pr \left( r_{p,t+1} < \mu_p - 1.96 \sigma_{p,t} \right) \]
Value at Risk

- **Synthetic Data** -
  - Four assets, (‘daily’ returns)
    - Each with a different regime switching volatility
    - Independent Regime Switching Correlation matrix
  - Assume constant weights, equally weighted, $w$
  - Calculate the next period VaR – assuming, $\alpha = 0.025$
Value at Risk

- Synthetic Data –

Total Price Level

Hidden Markov Chain: Correlation
Value at Risk

- Synthetic Data –
Value at Risk

- Using all data and true parameter values
Value at Risk

- Using a 2-year rolling window vs. truth
Value at Risk

- Using a 2 year rolling window and true parameter values
Portfolio Allocation

- What are the ‘best’ weights?
- Depends on utility function

\[
\begin{align*}
    u(r_{p,t+1}) &= w^T r_{t+1} - \lambda(w^T r_{t+1})^2 \\
    E[u(r_{p,t+1})] &= w^T \mu - \lambda w^T H_t w \\
    w &= \frac{2}{\lambda} (H_t^{-1}) \mu
\end{align*}
\]
Portfolio Allocation

- What are the ‘best’ weights?
- Depends on utility function

\[ w = \frac{2}{\lambda} (H_t^{-1}) \mu \]

<table>
<thead>
<tr>
<th></th>
<th>Optimal Weights Before Crisis</th>
<th>Optimal Weights After Crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>0.8615</td>
<td>0.6496</td>
</tr>
<tr>
<td>GOOG</td>
<td>-0.4703 (short)</td>
<td>-0.3079 (short)</td>
</tr>
<tr>
<td>GSK</td>
<td>-0.4523 (short)</td>
<td>0.3225</td>
</tr>
<tr>
<td>BOA</td>
<td>1.0610</td>
<td>0.3358</td>
</tr>
</tbody>
</table>
Portfolio Allocation

- How often should you change your weights?
- Transaction costs vs. changes in dynamics
Rebalancing Strategy

- Synthetic data set.
- Posterior Covariance calculated from our model.
- Sharpe Ratio.
- Re-balance frequency.

<table>
<thead>
<tr>
<th>Re-balance Strategy</th>
<th>Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Re-balance (3000)</td>
<td>0.168</td>
</tr>
<tr>
<td>Re-balance Every 44 Days (92)</td>
<td>0.116</td>
</tr>
<tr>
<td>Regime Switching Re-balance (91)</td>
<td>0.167</td>
</tr>
</tbody>
</table>

\[ SR_t = \frac{\mu_p}{\sigma_{p,t}} \]
Rebalancing Strategy

<table>
<thead>
<tr>
<th>Re-balancing Method</th>
<th>Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Re-balance (3000)</td>
<td>0.101</td>
</tr>
<tr>
<td>Re-balance Every 50 Days (60)</td>
<td>0.0924</td>
</tr>
<tr>
<td>Re-balance Every 40 Days (75)</td>
<td>0.0931</td>
</tr>
<tr>
<td>Regime Switching Re-balance (55)</td>
<td>0.0946</td>
</tr>
</tbody>
</table>

- World Indices data set.
- Posterior Covariance calculated from our model.
- Bayesian imputation Method.
Forward Looking Inference

• Historical time-series have limits
  – Identify past regimes
  – Provide confirmatory evidence that a new regime has started.

• Derivative prices are forward looking
  – Prices based on markets’ collective view of the future
  – Integrates over future price process (with an adjustment)
Forward Looking Inference

• Derivative prices are forward looking
  – Prices based on markets’ collective view of the future
  – Integrates over future price process (with an adjustment)
  – Derivative Price (Risk Neutral/State Price Measure, Q)

\[
C = E_Q [f(Z_T) | \mathcal{F}_t]
\]

\[
\text{Call}(Strike = K) = E_Q \left[ \max(Z_T - K, 0) | \mathcal{F}_t \right]
\]
Forward Looking Inference

• Derivative prices are forward looking
  – Prices based on markets’ collective view of the future
  – Integrates over future price process (with an adjustment)
  – Derivative Price (Risk Neutral/State Price Measure, Q)

\[ C = E_Q[f(Z_T)|\mathcal{F}_t] \]

– Key idea, if assuming Brownian Motion
– Physical and Risk Neutral measures have the same vol.

\[ d\ln(Z_t) = \mu_t \, dt + \sigma_t \, dW_t \quad \text{and} \quad d\ln(Z_t) = \mu_{RF} \, dt + \sigma_t \, d\tilde{W}_t \]
Forward Looking Inference

- Calibrate using historical data
Forward Looking Inference

- Calibrate using historical data
- Find best market implied, future volatility and correlation based on derivative prices and historical distributions.
Future Work

• Joint densities for all of the hidden Markov chains – finding common structure
• Better factor structures for high dimensional correlation matrices
• Better approximations for stochastic volatility option pricing formulas
• Extending framework to include Levy processes (jumps in price process)
• Testing the performance of risk measurement tools, using more realistic statistical models
• Inference for dynamic Empirical Pricing Kernel
Hidden Non Markov Chains

- Waiting time, not exponential
Natural Gas Example

![Graphs showing natural gas examples with Markovian and non-Markovian models.](image-url)

<table>
<thead>
<tr>
<th>Model</th>
<th>Harmonic Mean</th>
<th>Filter</th>
<th>DIC</th>
<th>Posterior Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markovian</td>
<td>-2764</td>
<td>*</td>
<td>5894</td>
<td>0</td>
</tr>
<tr>
<td>Non-Markovian</td>
<td>-2732</td>
<td>*</td>
<td>5850</td>
<td>1</td>
</tr>
</tbody>
</table>